Randomized Algorithms for Canonical Polyadic Decomposition

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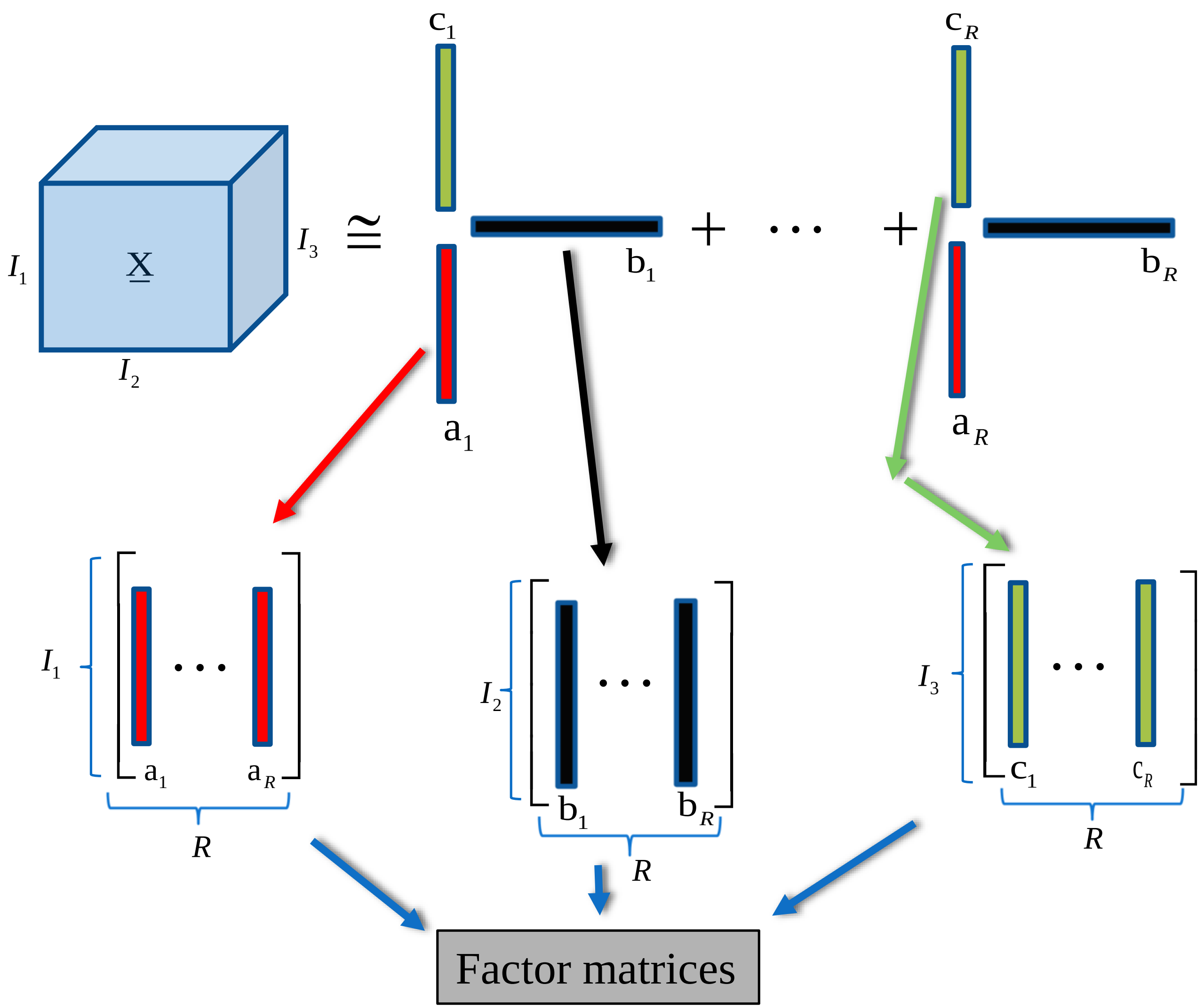
December 4, 2019

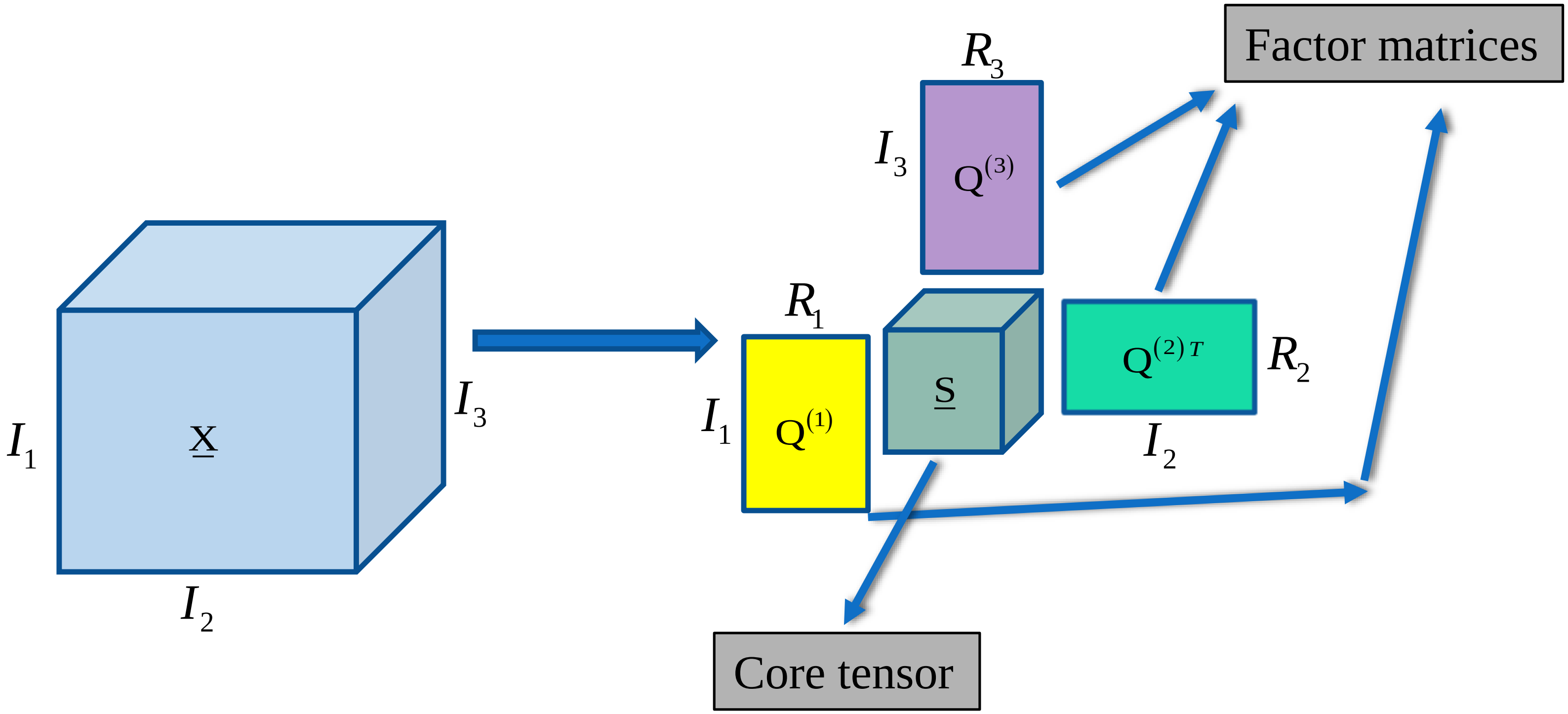
Outline

- **Canonical polyadic decomposition (CPD)**
- **Randomized algorithms for CPD**
  - Randomized CPD with prior reduction in Tucker format
  - Randomized block sampling algorithm
  - Randomized ALS algorithm
- **Randomized Parafac2**
- **Simulations**

Canonical Polyadic Decomposition (CPD)



Tucker decomposition



Orthogonal factor matrices and all-orthogonality of core tensor → HOSVD

Algorithms for CPD

- Alternating Least Squares (ALS) algorithm
- Gauss-Newton and Levenberg-Marquardt algorithms
- Tensor power iteration algorithm
- Fast damped Gauss-Newton (dGN)
- Simultaneous matrix diagonalization

Alternating least-squares algorithm (ALS)

Fixed all factor matrices and update one by solving the following least squats problem

$$\underset{\mathrm{A}^{(n)}}{\mathrm{Min}} \left\| Z^{(n)} \mathrm{A}^{(n)T} - \mathrm{X}_{(n)}^{T} \right\|_F , \; n = 1, 2, \ldots, N$$

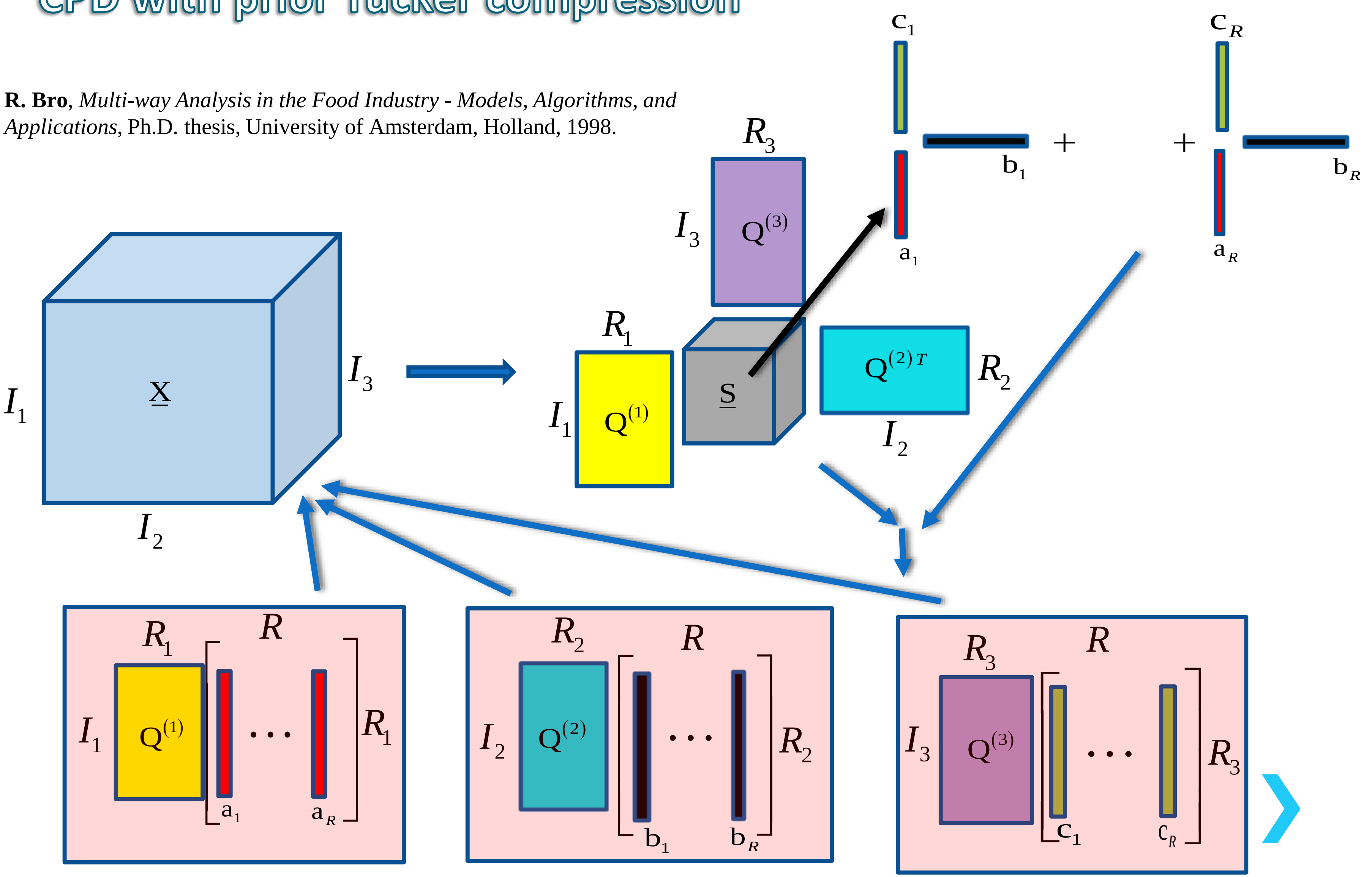
$$Z^{(n)} = \mathrm{A}^{(N)} \odot \cdots \odot \mathrm{A}^{(n+1)} \odot \mathrm{A}^{(n-1)} \cdots \odot \mathrm{A}^{(1)}$$

Explicit solution → $\mathrm{A}^{(n)} = \boxed{\mathrm{X}_{(n)} \mathrm{Z}^{(n)}} \, \mathrm{W}^{(n)\dagger}$

$$\mathrm{W}^{(n)} = \mathrm{A}^{(1)T}\mathrm{A}^{(1)} * \cdots * \mathrm{A}^{(n-1)T}\mathrm{A}^{(n-1)} * \mathrm{A}^{(n+1)T}\mathrm{A}^{(n+1)} * \cdots * \mathrm{A}^{(N)T}\mathrm{A}^{(N)}$$

CPD with prior Tucker compression

R. Bro, *Multi-way Analysis in the Food Industry - Models, Algorithms, and Applications*, Ph.D. thesis, University of Amsterdam, Holland, 1998.



CPD with prior HOSVD compression

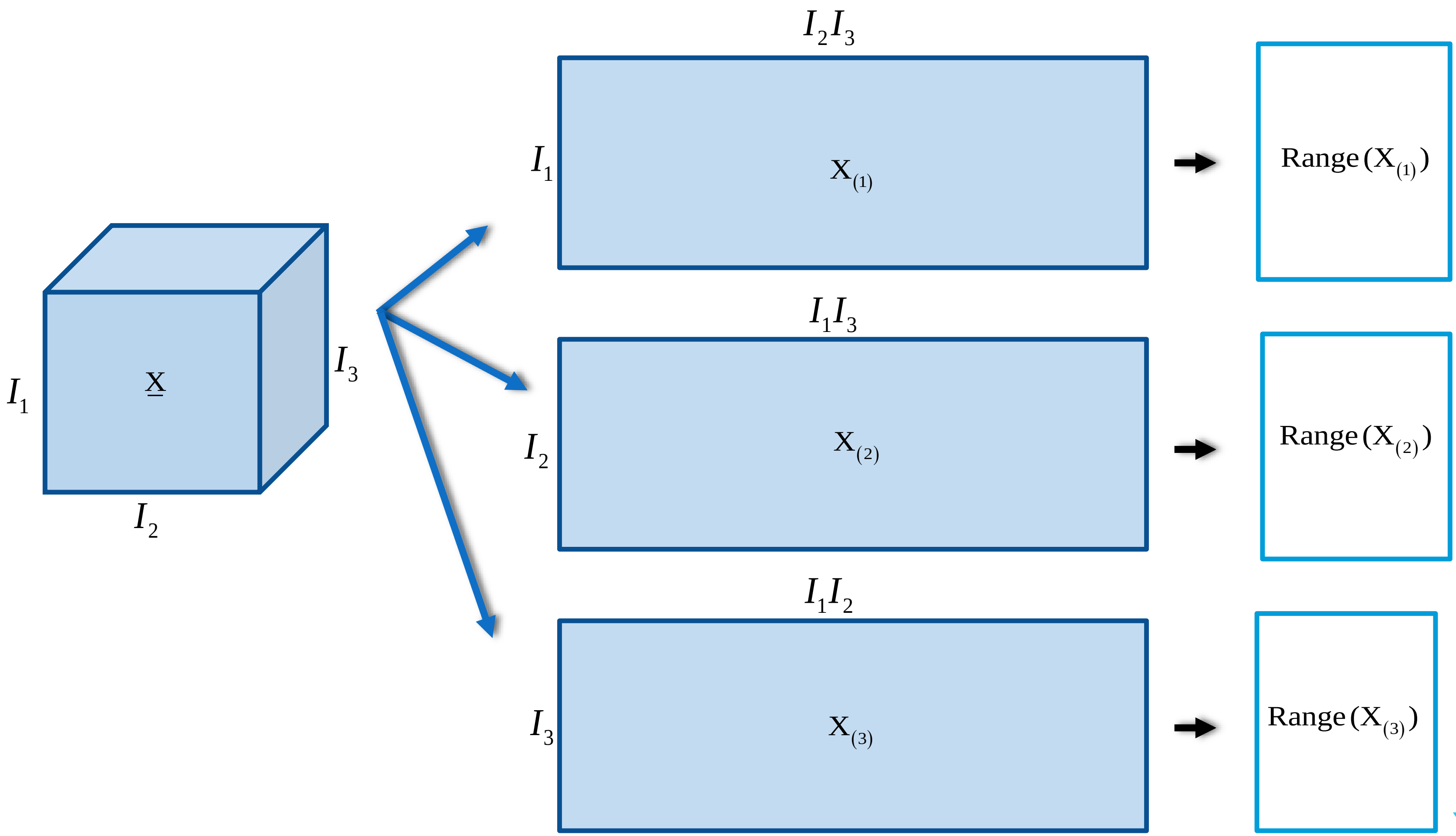
3-th order tensors $O\left(3R\prod_{i=1}^{3} I_i\right) \Rightarrow O\left(3R^4\right)$

N-th order tensors $O\left(NR\prod_{i=1}^{N} I_i\right) \Rightarrow O\left(NR^{N+1}\right)$

This approach is applicable when tensor is not of very high order and also the tensor rank is less than the original tensor sizes.

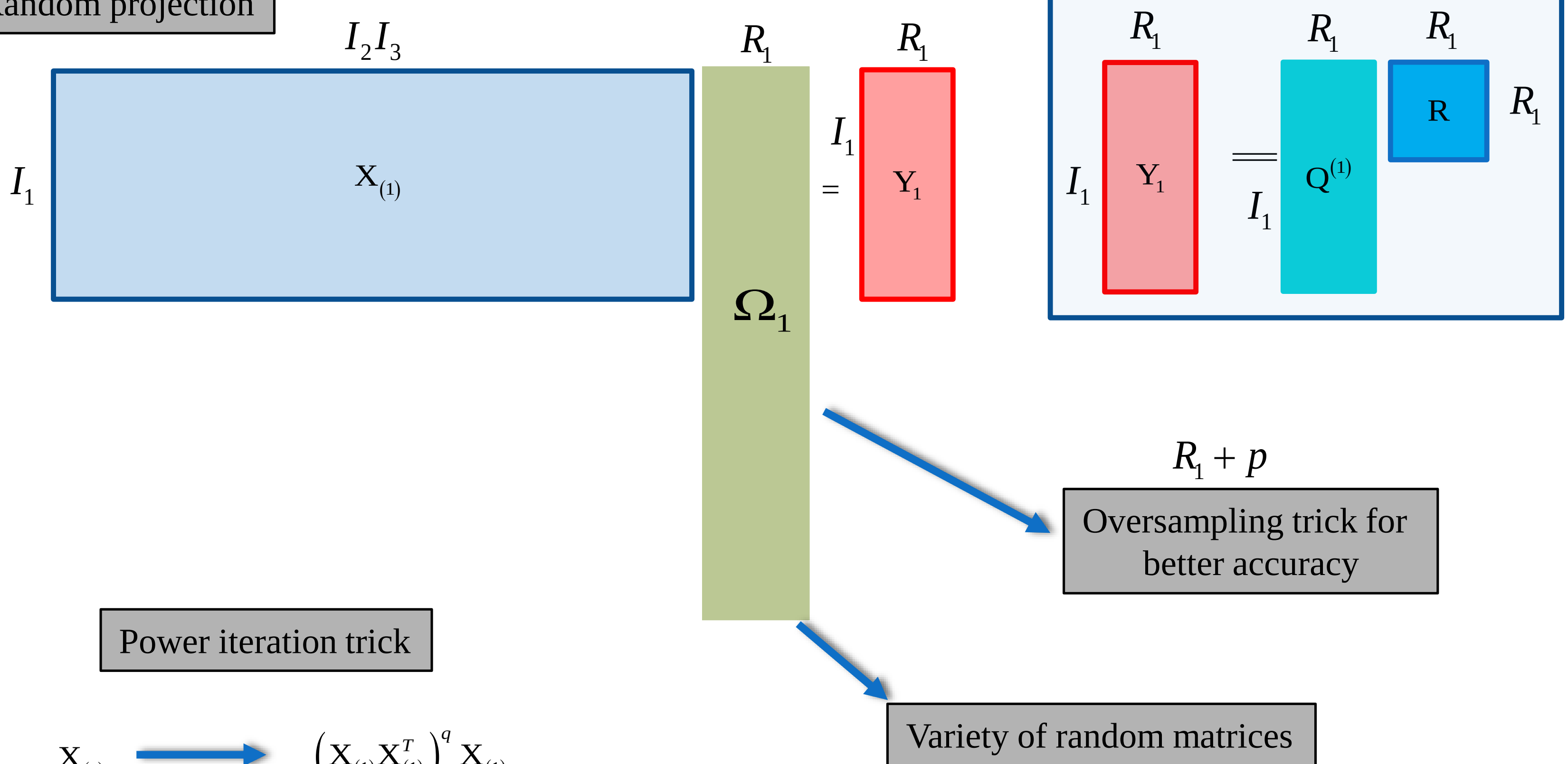
Compression step can be performed by randomized HOSVD algorithms

Randomized algorithms for HOSVD compression



Randomized algorithms for HOSVD compression

Random projection



$$X_{(1)} \longrightarrow \left(X_{(1)} X_{(1)}^T\right)^q X_{(1)}$$

Randomized algorithms for HOSVD compression

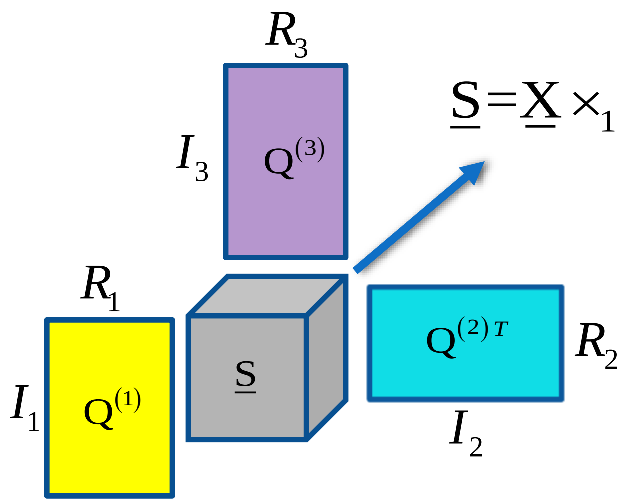
Sparse random matrices

$$\Phi = \begin{cases} -1 & \text{with probability } \dfrac{1}{6} \\ 0 & \text{with probability } \dfrac{2}{3} \\ +1 & \text{with probability } \dfrac{1}{6} \end{cases} \qquad \Phi = \begin{cases} -1 & \text{with probability } \dfrac{1}{2\sqrt{D}} \\ 0 & \text{with probability } 1-\dfrac{1}{\sqrt{D}} \\ +1 & \text{with probability } \dfrac{1}{2\sqrt{D}} \end{cases}$$

where matrix components are i.i.d.

These random matrices are reminiscent the quantization procedure in learning DNNs.

Randomized algorithms for HOSVD compression

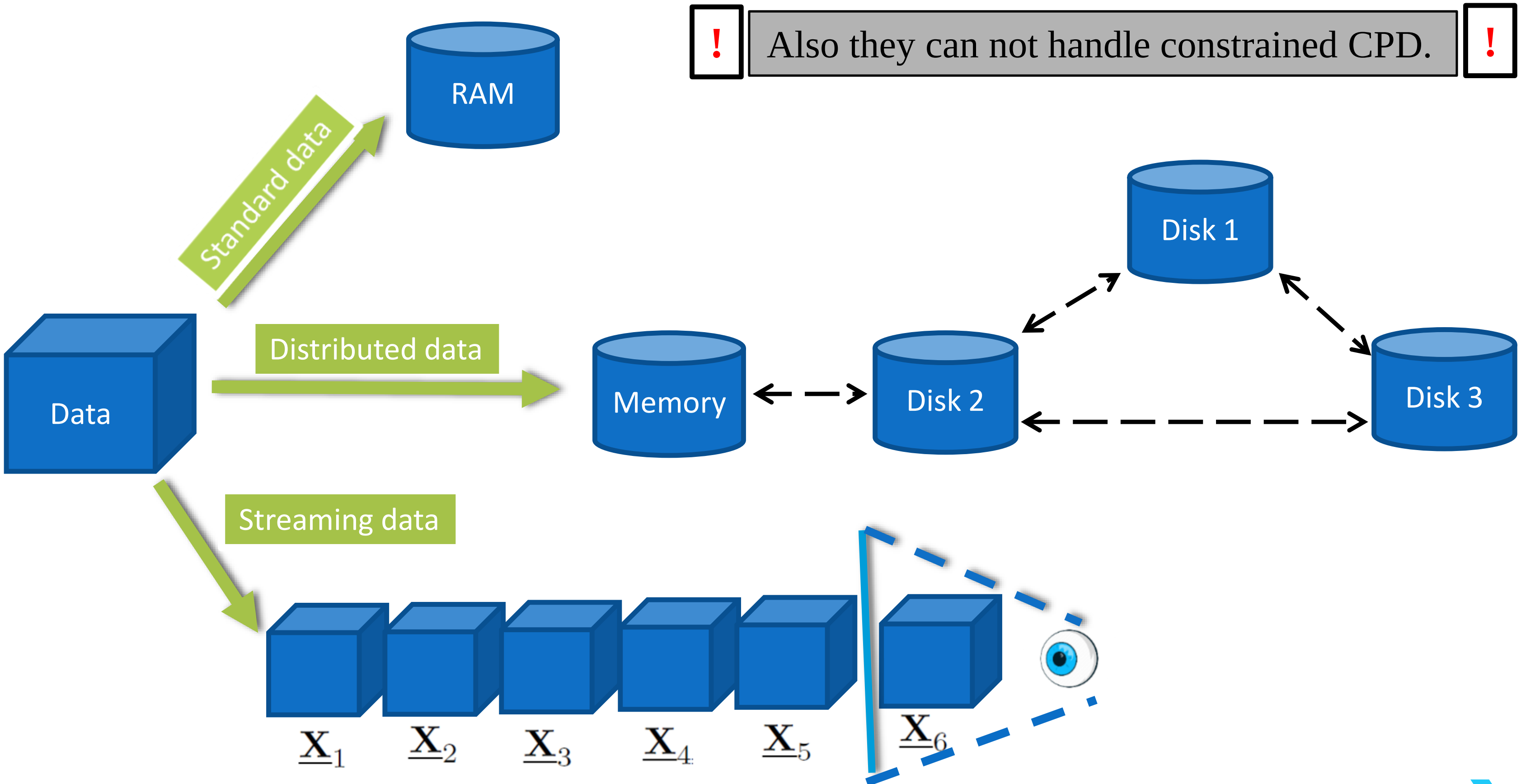


$$\underline{S} = \underline{X} \times_1 Q^{(1)\,T} \times_2 Q^{(2)\,T} \times_3 Q^{(3)\,T}$$

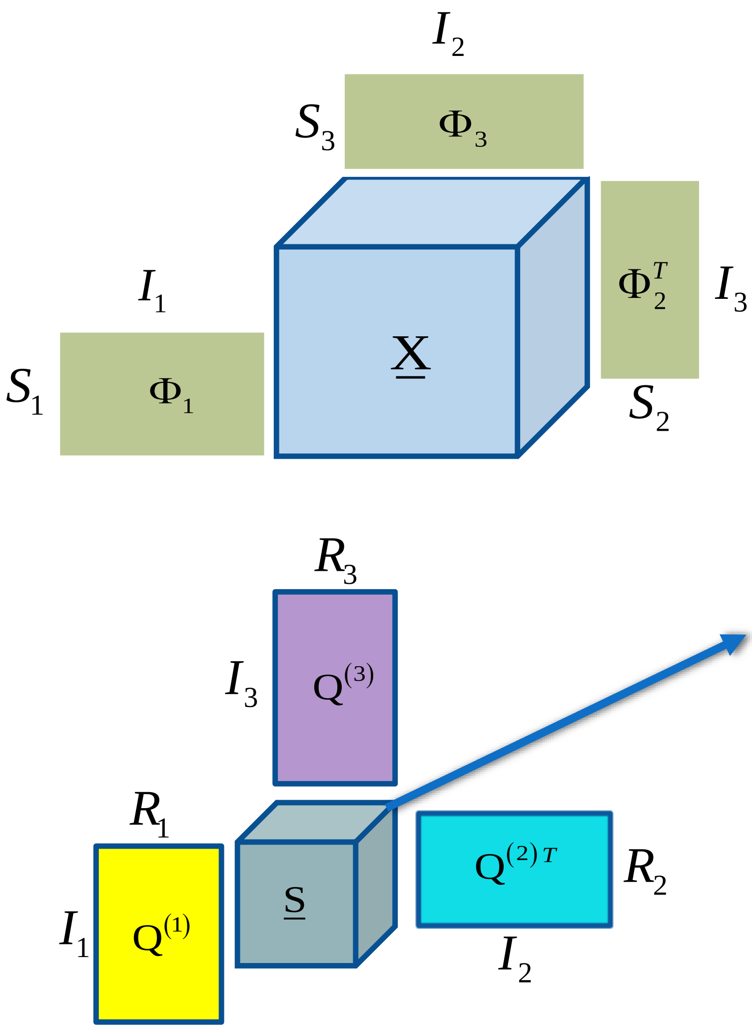
Different types of data

Algorithms presented in this talk are well-suite for first and second ones.

! Also they can not handle constrained CPD. !



Randomized algorithms for HOSVD compression

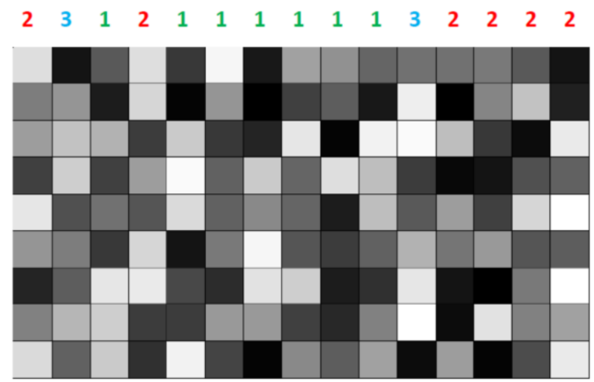


$$\underline{S} \cong \underline{H} \times_1 \left(\Phi_1 Q^{(1)}\right)^{\dagger} \times_2 \left(\Phi_2 Q^{(2)}\right)^{\dagger} \times_3 \left(\Phi_3 Q^{(3)}\right)^{\dagger}$$

Randomized algorithms for HOSVD compression

Count sketch

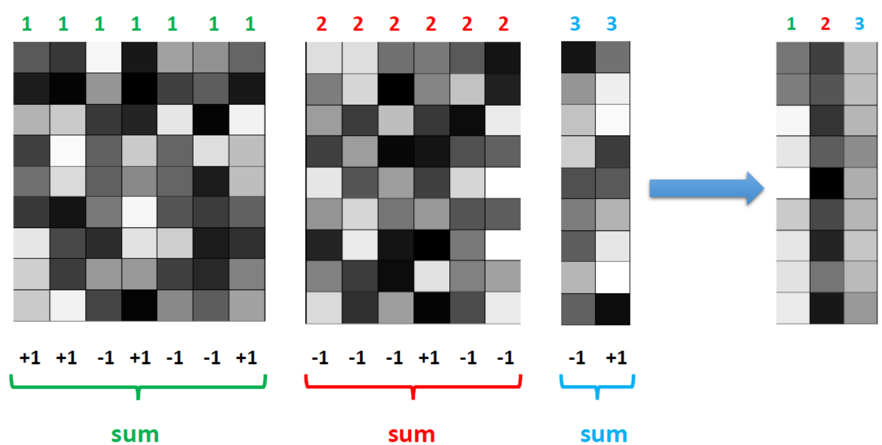
Pictures from:
S. Wang. *A practical guide to randomized matrix computations with MATLAB implementations*. arXiv:1505.07570, 2015.



Example:
- Matrix size 9×15
- Sketch size $s = 3$

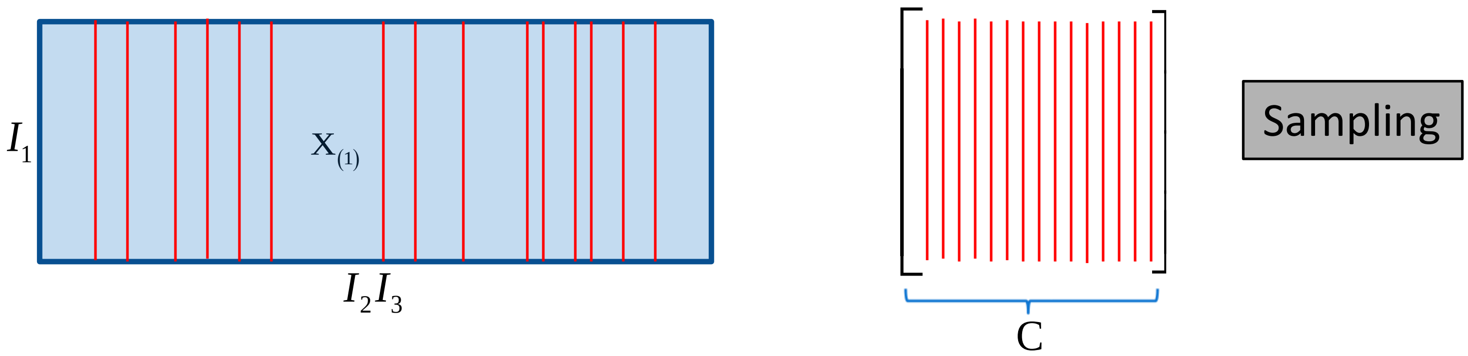
Hashing

(a) Hash each column with a value uniformly sampled from $[s] = \{1, 2, 3\}$.



(b) Flip the sign of each column with probability 50%, and then sum up columns with the same hash value.

Randomized algorithms for HOSVD compression



Sampling columns

- Uniform sampling (with or without replacement)
- Length-squared distribution
- Leverage scores distribution

Randomized algorithms for HOSVD compression

$$\mathbf{X}_{(n)} \in \mathbb{R}^{I_n \times \prod_{m \neq n} I_m}$$

j-th column leverage score

$$l_j = \left\| \mathrm{V}_R(j,:) \right\|_2^2, \quad j = 1, 2, \ldots, \prod_{m \neq n} I_m$$

$$\sum_{j=1}^{\prod_{m \neq n} I_m} l_j = R$$

i-th row leverage score

$$l_i = \left\| \mathrm{U}_R(i,:) \right\|_2^2, \quad i = 1, 2, \ldots, I_n$$

$$\sum_{j=1}^{I_n} l_i = R$$

Leverage score distribution for column selection

$$p_j = \frac{l_j}{R}, \qquad j = 1, 2, \ldots, \prod_{m \neq n} I_m$$

Maximum of leverage score is called coherence of a matrix

Randomized algorithms for HOSVD compression

Fast computation of leverage scores are proposed in the following papers.

Journal of Machine Learning Research 13 (2012) 3441-3472 Submitted 7/12; Published 12/12

Fast Approximation of Matrix Coherence and Statistical Leverage

Petros Drineas DRINEP@CS.RPI.EDU
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Stanford University
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David P. Woodruff DPWOODRU@US.IBM.COM
IBM Almaden Research Center
650 Harry Road
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Editor: Mehryar Mohri

On Fast Leverage Score Sampling and Optimal Learning

Alessandro Rudi* INRIA – Sierra team, ENS, Paris
Daniele Calandriello* LCSL – IIT & MIT, Genoa, Italy
Luigi Carratino University of Genoa, Genoa, Italy
Lorenzo Rosasco University of Genoa, LCSL – IIT & MIT

Abstract

Leverage score sampling provides an appealing way to perform approximate computations for large matrices. Indeed, it allows to derive faithful approximations with a complexity adapted to the problem at hand. Yet, performing leverage scores sampling is a challenge in its own right requiring further approximations. In this paper, we study the problem of leverage score sampling for positive definite matrices defined by a kernel. Our contribution is twofold. First we provide a novel algorithm for leverage score sampling and second, we exploit the proposed method in statistical learning by deriving a novel solver for kernel ridge regression. Our main technical contribution is showing that the proposed algorithms are currently the most efficient and accurate for these problems.

Randomized algorithms for HOSVD compression

$$\|\mathrm{X} - \mathrm{C}\mathrm{C}^{\dagger}\mathrm{X}\| \le \varepsilon \|\mathrm{X} - \mathrm{X}_R\|$$

Relative error

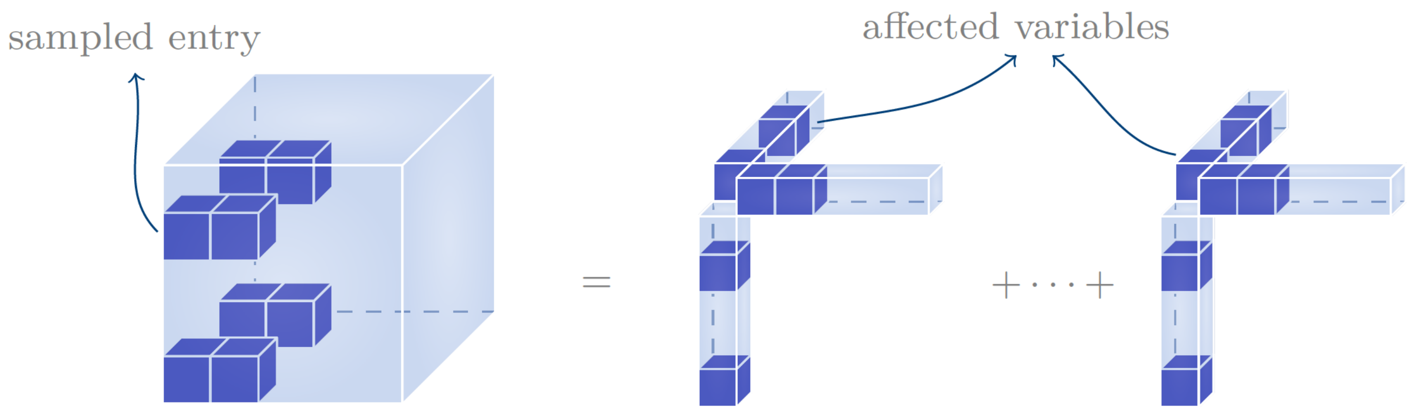
$$\|\mathrm{X} - \mathrm{C}\mathrm{C}^{\dagger}\mathrm{X}\| \le \|\mathrm{X} - \mathrm{X}_R\| + \varepsilon \|\mathrm{X}_R\|$$

Additive error

Randomized rank revealing or equivalently randomized fixed-precision algorithms are applicable when an estimation of matrix rank is unknown and it is approximated adaptively by the underlying algorithms.

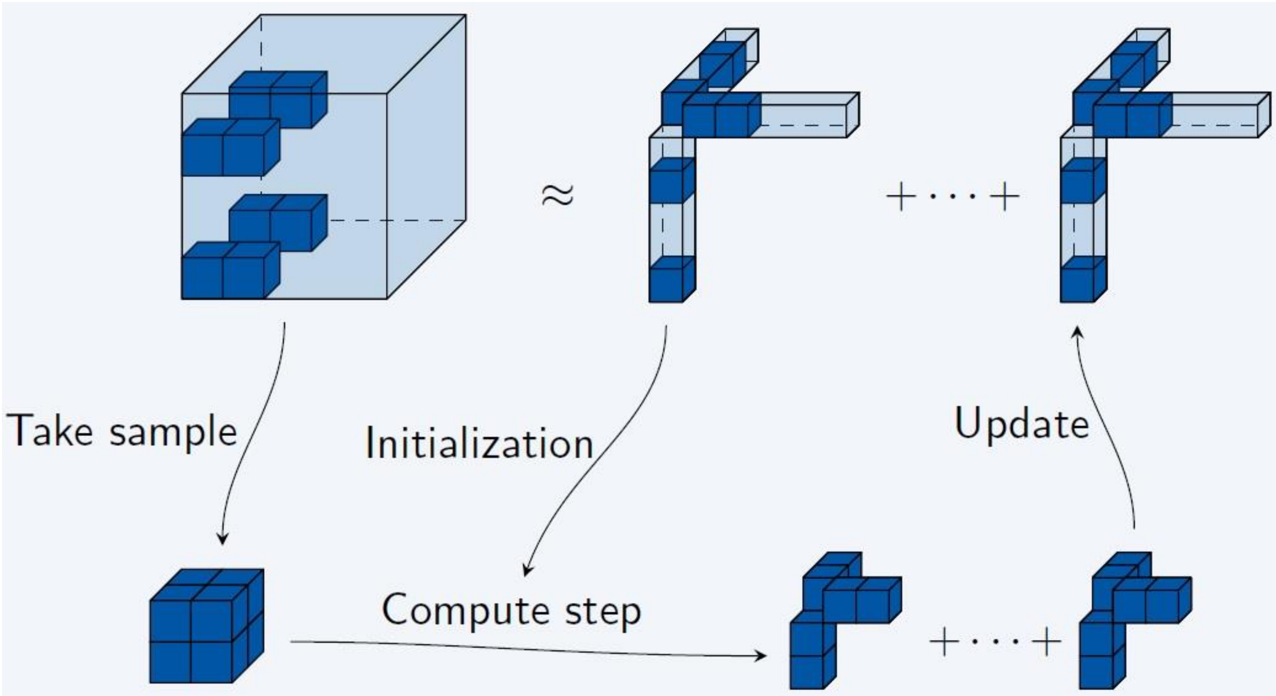
Randomized block sampling algorithm

Randomized block sampling uses the locality property to compute CPD of a given matrix.



Picture from: N. Vervliet, *Compressed sensing approaches to large-scale tensor decompositions*, PhD thesis, KU Leuven, May 2018.

Randomized block sampling algorithm

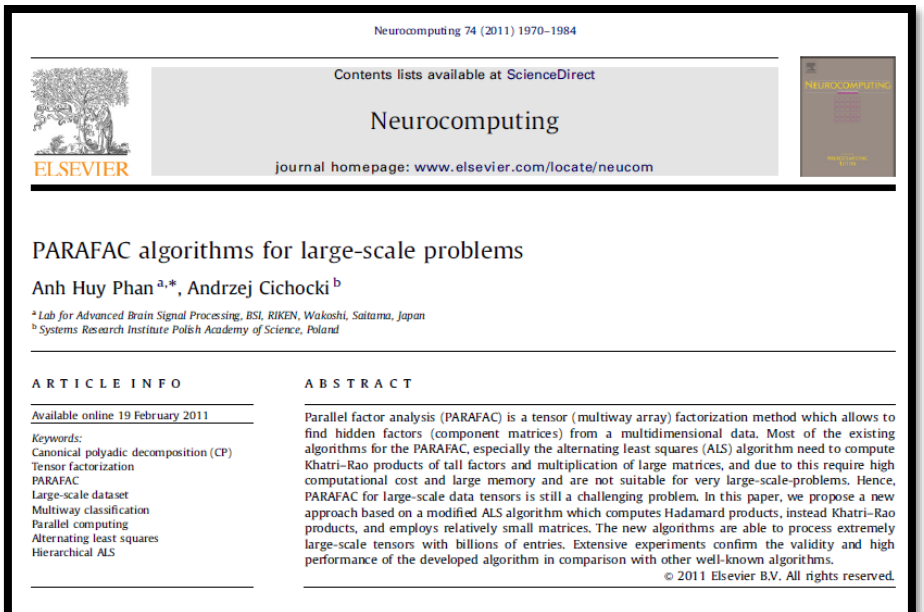


Picture from: N. Vervliet, *Compressed sensing approaches to large-scale tensor decompositions*, PhD thesis, KU Leuven, May 2018.

Randomized block sampling algorithm

Matlab codes are accessible in tensorlab toolbox

N. Vervliet, O. Debals, L. Sorber, M. V. Barel, L. D. Lathauwer, Tensorlab 3.0.



Neurocomputing 74 (2011) 1970–1984

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journal homepage: www.elsevier.com/locate/neucom

PARAFAC algorithms for large-scale problems

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Available online 19 February 2011

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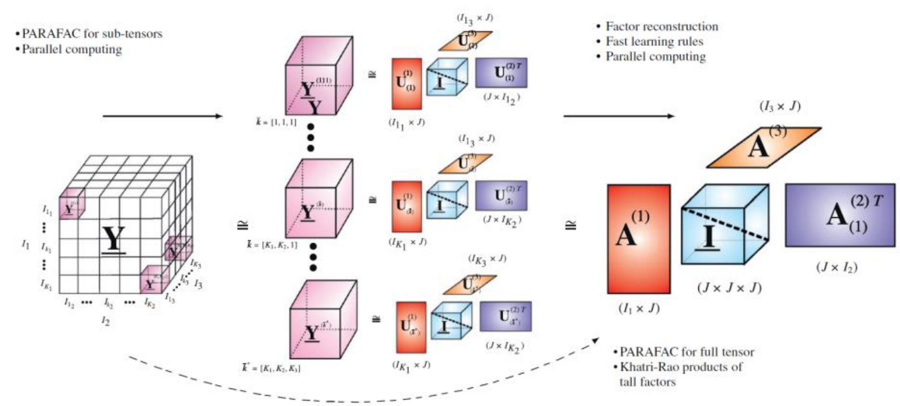
PARAFAC
Large-scale dataset
Multiway classification
Parallel computing
Alternating least squares
Hierarchical ALS

ABSTRACT

Parallel factor analysis (PARAFAC) is a tensor (multiway array) factorization method which allows to find hidden factors (component matrices) from a multidimensional data. Most of the existing algorithms for the PARAFAC, especially the alternating least squares (ALS) algorithm need to compute Khatri–Rao products of tall factors and multiplication of large matrices, and due to this require high computational cost and large memory and are not suitable for very large-scale-problems. Hence, PARAFAC for large-scale data tensors is still a challenging problem. In this paper, we propose a new approach based on a modified ALS algorithm which computes Hadamard products, instead Khatri–Rao products, and employs relatively small matrices. The new algorithms are able to process extremely large-scale tensors with billions of entries. Extensive experiments confirm the validity and high performance of the developed algorithm in comparison with other well-known algorithms.

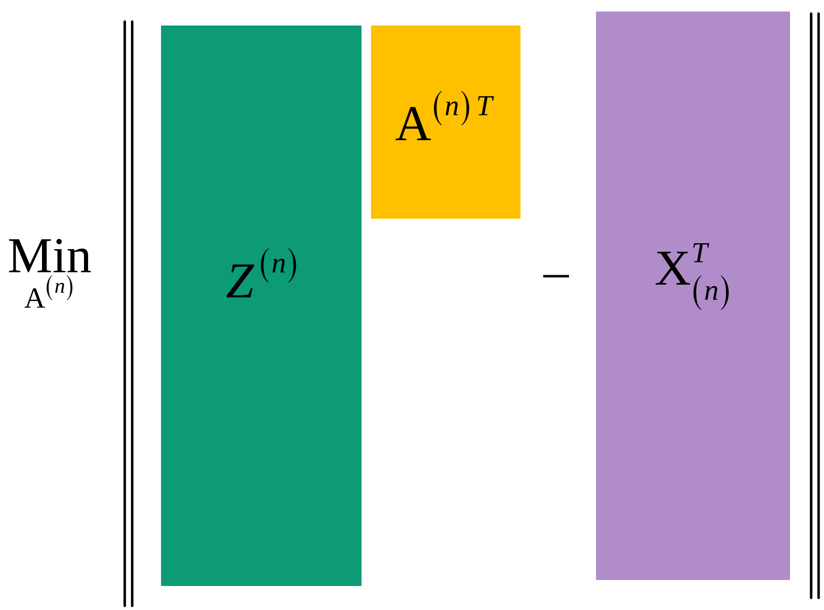


Pictures from

Alternating randomized least-squares algorithms

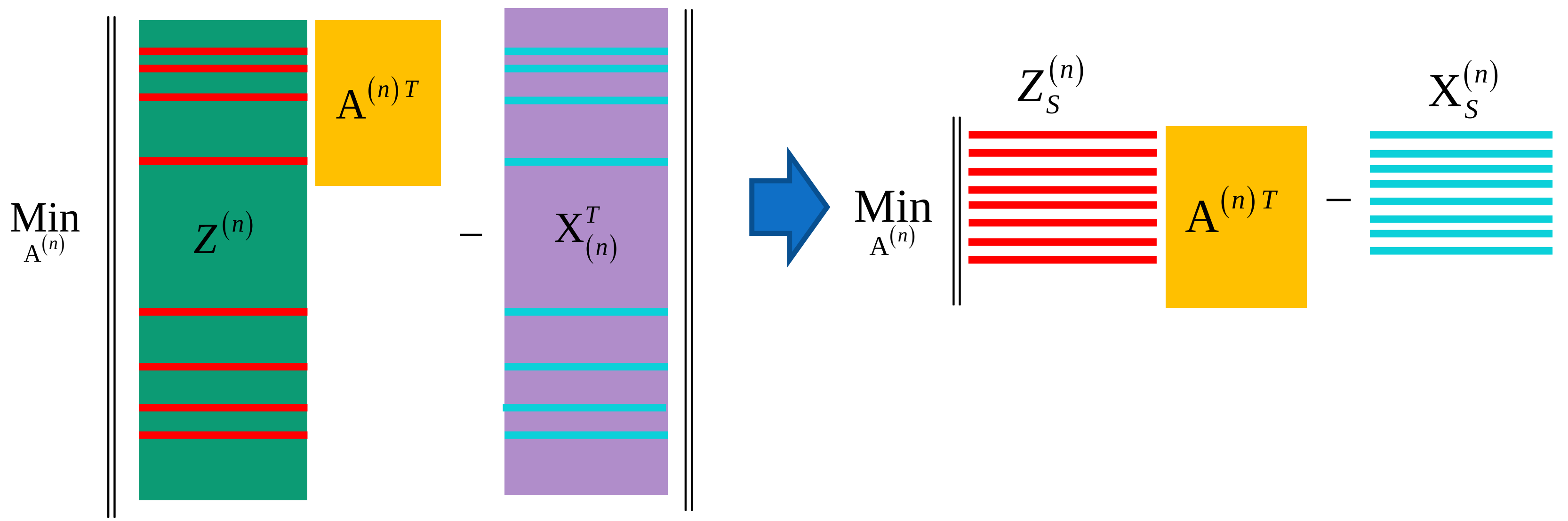
$$\underset{\mathrm{A}}{\mathrm{Min}} \left\| \mathrm{Z}^{(n)} \mathrm{A}^{(n)T} - \mathrm{X}_{(n)}^{T} \right\|_F , \; n = 1, 2, \ldots, N$$

$$Z^{(n)} = \mathrm{A}^{(N)} \odot \cdots \odot \mathrm{A}^{(n+1)} \odot \mathrm{A}^{(n-1)} \cdots \odot \mathrm{A}^{(1)}$$



$$\mathrm{A}^{(n)} = \boxed{\mathrm{X}_{(n)} \mathrm{Z}^{(n)}} \, \mathrm{W}^{(n)\dagger}$$

Alternating randomized least-squares algorithms

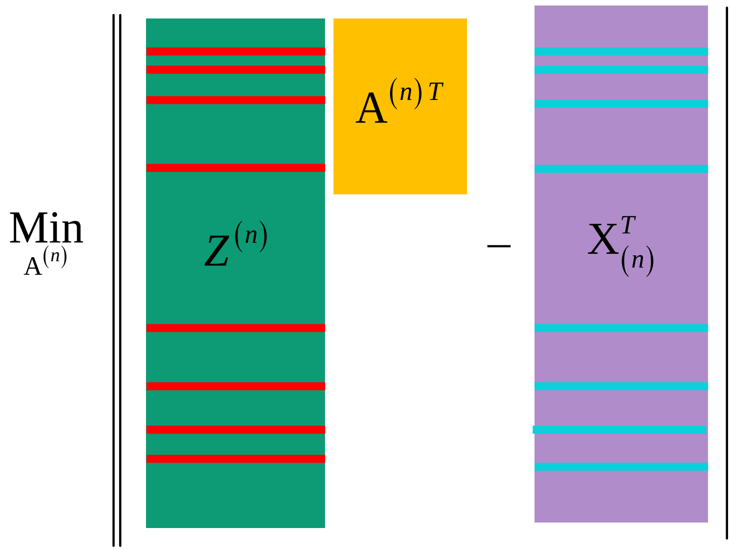


Sampling

- Uniform sampling
- Nonuniform sampling

Coherence of matrix should be as small as possible

Alternating randomized least-squares algorithms



The column selection does not need to compute $Z^{(n)}$ explicitly.

$$Z^{(n)}(j,:) = A^{(1)}(i_1,:) * \cdots * A^{(n-1)}(i_{n-1},:) * A^{(n+1)}(i_{n+1},:) * \cdots * A^{(N)}(i_N,:)$$

$$j = 1 + \sum_{k=1,k\neq n}^{N} (i_k - 1) J_k, \quad J_k = \prod_{m=1,m\neq n}^{N-1} I_m$$

Alternating randomized least-squares algorithms

Sampling

- Uniform sampling → Coherence of matrix should be as small as possible
- Nonuniform sampling (Leverage scores)

SPALS: Fast Alternating Least Squares via Implicit Leverage Scores Sampling

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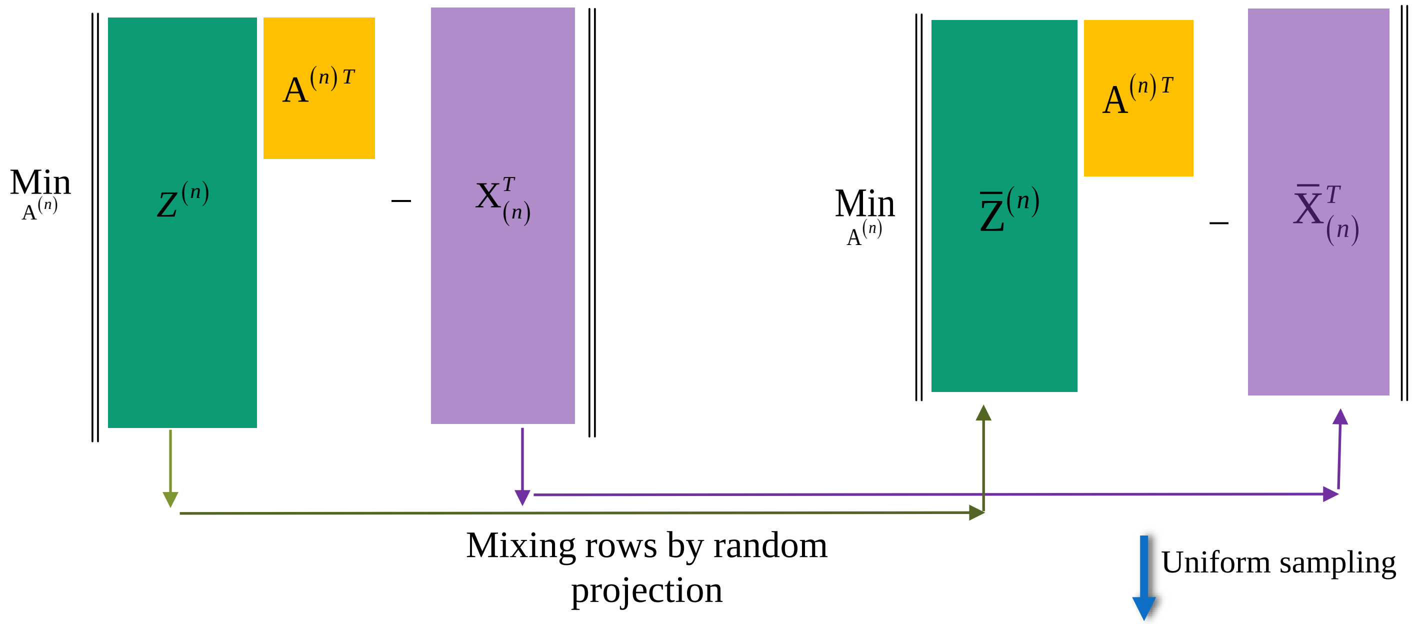
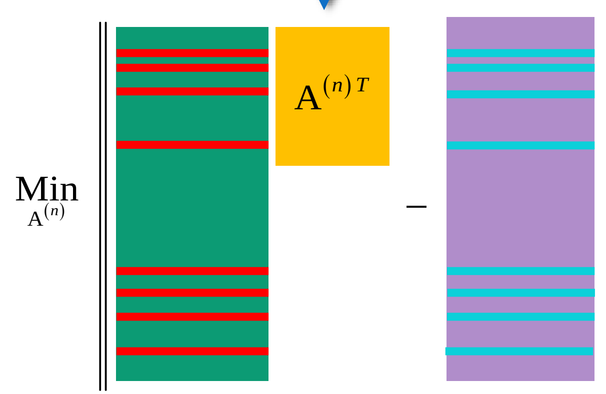
For both Khatri-Rao and Kroncker product of matrices

Alternating randomized least-squares algorithms





C. Battaglino, G. Ballard, T. G. Kolda, A practical randomized cp tensor decomposition, SIAM Journal on Matrix Analysis and Applications 39 (2) (2018) 876–901.

Alternating randomized least-squares algorithms

Matlab codes are included in tensor toolbox (Matlab)

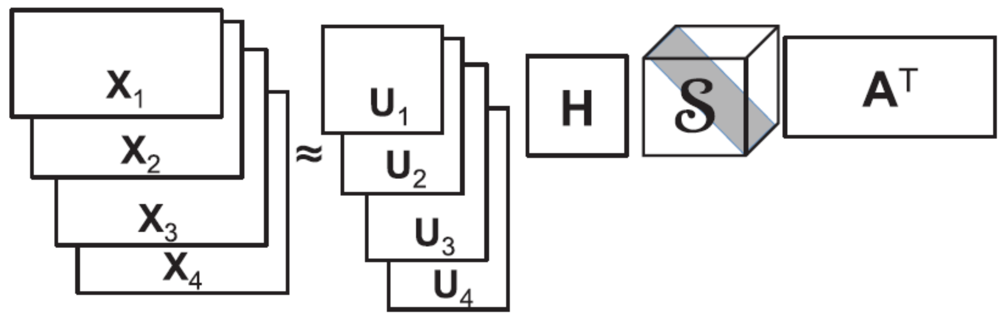
B. W. Bader, T. G. Kolda, Matlab tensor toolbox version 2.6 (2015).

Matlab codes are included in Tensorly (Python)

J. Kossaifi, Y. Panagakis, A. Anandkumar, M. Pantic, Tensorly: *Tensor learning in python*, The Journal of Machine Learning Research 20 (1) (2019) 925–930.

Randomized algorithms for parafac2

Parafac2 model



Picture from: E.E. Papalexakis, C. Faloutsos, N.D. Sidiropoulos, *Tensors for data mining and data fusion: Models, applications, and scalable algorithms*, ACM Trans. on Intelligent Systems and Technology, vol. 8, no. 2, 16:1–16:44, 2016.

Randomized algorithms for parafac2

ALGORITHM 8: ALS Algorithm for PARAFAC2

Input: Multiset $\{\mathbf{X}_k\}$ for $k = 1 : K$ and rank R.
Output: PARAFAC2 Decomposition of $\{\mathbf{X}_k\}$: $\{\mathbf{U}_k\}$, $\mathbf{H}$, $\mathcal{S}$, $\mathbf{V}$.

Initialize:

$$\mathbf{V} \leftarrow R \text{ principal eigenvectors of } \sum_{k=1}^{K} \mathbf{X}_k^T \mathbf{X}_k$$

$\mathbf{H} \leftarrow \mathbf{I}$

for $k = 1 \cdots K$ **do**
  $\mathcal{S}(:,:,k) \leftarrow \mathbf{I}$, for $k = 1 \cdots K$.
end for
while convergence criterion is not met **do**
  for $k = 1 \cdots K$ **do**
    $[\mathbf{P}_k, \mathbf{\Sigma}_k, \mathbf{Q}_k] \leftarrow$ truncated SVD of $\mathbf{H}\mathbf{S}_k\mathbf{V}^T\mathbf{X}_k^T$ at rank R
    $\mathbf{U}_k \leftarrow \mathbf{Q}_k\mathbf{P}_k^T$
  end for
  for $k = 1 \cdots K$ **do**
    Compute $\mathcal{Y}(:,:,k) = \mathbf{U}_k^T\mathbf{X}_k$
  end for
  Run a single iteration of CP ALS (Algorithm 1) on $\mathcal{Y}$ and compute factors $\mathbf{H}$, $\mathbf{V}$, $\hat{\mathbf{S}}$.
  for $k = 1 \cdots K$ **do**
    $\mathcal{S}(:,:,k) \leftarrow Diag(\hat{\mathbf{S}}(k,:))$
  end for
end while

Variety of Randomized algorithms can be used.

Picture from: E.E. Papalexakis, C. Faloutsos, N.D. Sidiropoulos, *Tensors for data mining and data fusion: Models, applications, and scalable algorithms*, ACM Trans. on Intelligent Systems and Technology, vol. 8, no. 2, 16:1–16:44, 2016.

Simulations

Collinearity

$[0.7, .8]$

$$\mathbf{A}^{(1)} = \mathrm{randn}\,(500, R)\,, \quad \mathbf{A}^{(2)} = \mathrm{randn}\,(500, R)\,,$$

$$\mathbf{A}^{(3)} = \mathrm{randn}\,(500, R)\,.$$

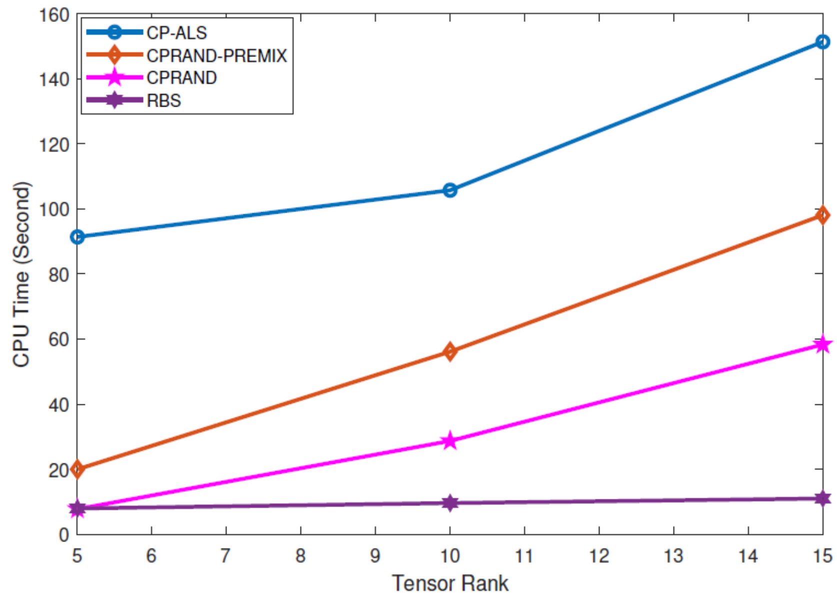
$\eta = 0.01$

$$\underline{\mathbf{X}} = [[\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \mathbf{A}^{(3)}]] \qquad \widehat{\underline{\mathbf{X}}} = \underline{\mathbf{X}} + \eta \left(\frac{\|\underline{\mathbf{X}}\|}{\|\underline{\mathbf{Y}}\|} \right) \underline{\mathbf{Y}},$$

where $\underline{\mathbf{Y}}$ is a 4-th order tensor with i.i.d Gaussian components.

$$f = 1 - \frac{\left\| \widehat{\underline{\mathbf{X}}} - \widetilde{\underline{\mathbf{X}}} \right\|}{\left\| \widehat{\underline{\mathbf{X}}} \right\|}$$

Simulations

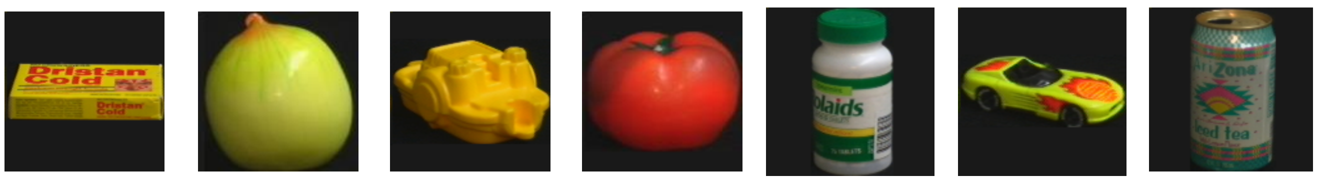


Algorithm	$R = 5$	$R = 10$	$R = 15$
CP-ALS	0.9900	0.9900	0.9900
CPRAND-PREMIX	0.9897	0.9898	0.9898
CPRAND	0.9893	0.9895	0.9896
RBS	0.9881	0.9897	0.9898

Simulations

Coil-100 dataset

S. Nene, S. Nayar, and H. Murase, Columbia Object Image Library (COIL-100), Tech. Report CUCS-006-96, Columbia University, 1996.



100 different object classes

72 different angles

$$128 \times 128 \times 3 \times 7200$$

Simulations

For CP-rank $R = 20$ the following results were achieved

	Cpu Time (Second)	Fit
CP-ALS	220	0.686
CPRAND-PREMIX	77.46	0.684

Simulations

Video compression

Video link: https://www.youtube.com/watch?v=1qeWugmiGt4

$\text{Time duration}: \ 1'':29'''$

$\text{size}: 360 \times 480 \times 3 \times 2200$

$360 \times 480 \times 3 \times 300$

Simulations

Frame number: 1

Frame number: 20

Frame number: 60

Original video

Frame number: 100

Compressed video by CP rank

$R = 100$

Here again CP-PREMIX was approximately 2 times faster than CP-ALS.

Thanks for your attention!